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PAULI-VILLARS REGULARIZATION AND DISCRETE LIGHT-CONE QUANTIZATION IN YUKAWA THEORY[a]

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The techniques of Pauli–Villars regularization and discrete light-cone quantization are combined to analyze Yukawa theory in a single-fermion truncation. A special form of the Lanczos algorithm is constructed for diagonalization of the indefinite-metric light-cone Hamiltonian.

1 Introduction

As a step toward development of a method for nonperturbative solution of four-dimensional quantum field theories, we consider a single-fermion truncation of Yukawa theory in discrete light-cone quantization (DLCQ). [1] To regulate the theory we introduce Pauli-Villars (PV) bosons with indefinite metric [2] into the Fock basis. This extends earlier work on model theories by Brodsky, Hiller and McCartor [3,4] to a more physical situation.

The importance of Pauli–Villars regularization stems from its ability to regulate the continuum theory with counterterms generated automatically, except for a trivial mass counterterm. The numerical method, in this case DLCQ, can then be applied to a finite theory. The bare parameters are fixed via physical constraints and become functions of the PV masses and of any numerical parameters. The original theory is recovered in the following sequence of limits: infinite numerical resolution, infinite (momentum) volume, and infinite PV masses.

The DLCQ formulation is based on periodic boundary conditions for bosons and antiperiodic conditions for fermions in a light-cone box $-L < x^- \equiv (t - z) < L$, $-L_\perp < x, y < L_\perp$. The light-cone 3-momentum $\underline{p} \equiv (p^+, \vec{p}_\perp)$ is then on a discrete grid specified by integers $\underline{n} = (n, n_x, n_y)$: $p^+ \equiv (E + p_z) = \frac{\pi}{L} n$, $\vec{p}_\perp = (\frac{\pi}{L_\perp} n_x, \frac{\pi}{L_\perp} n_y)$. The limit $L \to \infty$ can be exchanged for a limit in terms

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of the integer *resolution* [5] $K \equiv \frac{L}{\pi}P^+$ for total longitudinal momentum P^+. Also $x \equiv p^+/P^+$ becomes n/K, with n odd for fermions and even for bosons. The light-cone Hamiltonian $H_{\rm LC} \equiv P^+P^-$ is independent of L.

Because each n is positive, DLCQ automatically limits the number of particles to no more than $\sim K/2$. The integers n_x and n_y range between limits associated with some maximum integer $N_\perp$ fixed by the invariant-mass cutoff $m_i^2 + p_{\perp i}^2 \leq x_i\Lambda^2$ for each particle i. We then have a finite matrix representation where integrals are replaced by discrete sums

$$\int dp^+ \int d^2p_\perp f(p^+,\vec{p}_\perp) \simeq \frac{2\pi}{L}\left(\frac{\pi}{L_\perp}\right)^2 \sum_{\underline{n}} f(n\pi/L, \vec{n}_\perp\pi/L_\perp)\,. \tag{1}$$

This trapezoidal approximation is improved through the inclusion of weighting factors [3] that take into account distances between grid point locations and the integration boundaries set by the cutoff Λ^2. In the following sections we discuss how these techniques can be applied to Yukawa theory.

2 Yukawa theory

The DLCQ Hamiltonian for Yukawa theory, when truncated to include only one fermion, is [6]

$$\begin{aligned}
H_{\rm LC} = &\sum_{\underline{n},s} \frac{M^2+\delta M^2+(\vec{n}_\perp\pi/L_\perp)^2}{n/K} b_{\underline{n},s}^\dagger b_{\underline{n},s} + \sum_{\underline{m}i} \frac{\mu_i^2+(\vec{m}_\perp\pi/L_\perp)^2}{m/K} a_{i\underline{m}}^\dagger a_{i\underline{m}} \\
&+\frac{g\sqrt{\pi}}{2L_\perp^2}\sum_{\underline{nm}}\sum_{si}\frac{\xi_i}{\sqrt{m}}\left(\left[\frac{\vec{\epsilon}^{\,*}_{-2s}\cdot\vec{n}_\perp}{n/K}+\frac{\vec{\epsilon}_{2s}\cdot(\vec{n}_\perp+\vec{m}_\perp)}{(n+m)/K}\right] b_{\underline{n}+\underline{m},-s}^\dagger b_{\underline{n},s} a_{i\underline{m}} + \text{h.c.}\right) \\
&+\frac{Mg}{\sqrt{8\pi}L_\perp}\sum_{\underline{nm}}\sum_{si}\frac{\xi_i}{\sqrt{m}}\left(\left[\frac{1}{n/K}+\frac{1}{(n+m)/K}\right] b_{\underline{n}+\underline{m},s}^\dagger b_{\underline{n},s} a_{i\underline{m}} + \text{h.c.}\right) \qquad (2) \\
&+\frac{g^2}{8\pi L_\perp^2}\sum_{\underline{nmm'}}\sum_{sij}\frac{\xi_i\xi_j}{\sqrt{mm'}}\left[\left(b_{\underline{n}+\underline{m}+\underline{m}',s}^\dagger b_{\underline{n},s} a_{i\underline{m}'} a_{j\underline{m}}\frac{1}{(n+m)/K} + \text{h.c.}\right)\right. \\
&\qquad\left. + b_{\underline{n}+\underline{m}-\underline{m}',s}^\dagger b_{\underline{n},s} a_{i\underline{m}'}^\dagger a_{j\underline{m}}\left(\frac{1}{(n-m')/K}+\frac{1}{(n+m)/K}\right)\right],
\end{aligned}$$

where M is the fermion mass, $\mu \equiv \mu_0$ the physical boson mass, μ_i the ith PV boson mass, $\vec{\epsilon}_\lambda = -\frac{1}{\sqrt{2}}(\lambda, i)$ the polarization vector for helicity λ, and

$$\left[a_{i\underline{m}}, a_{j\underline{m}'}^\dagger\right] = \delta_{ij}\delta_{\underline{m},\underline{m}'}\,, \quad \left\{b_{\underline{n},s}, b_{\underline{n}',s'}^\dagger\right\} = \delta_{\underline{n},\underline{n}'}\delta_{s,s'}\,. \tag{3}$$

Modes with zero longitudinal momentum have not been included. Fermion self-induced inertia terms are also not included, because they cancel between PV-boson terms. A fermion mass counterterm has been included to remove shifts proportional to $\ln \mu_i/\mu$.

The number of PV flavors is three. Their couplings are given by $\xi_i g$, where $\xi_i = \sqrt{|C_i|}$ and

$$1 + \sum_{i=1}^{3} C_i = 0\,, \quad \mu^2 + \sum_{i=1}^{3} C_i \mu_i^2 = 0\,, \quad \sum_{i=1}^{3} C_i \mu_i^2 \ln(\mu_i^2/\mu^2) = 0\,. \tag{4}$$

The sign of C_i determines the norm. This arrangement is known[7,3] to produce the cancellations needed to regulate the theory and restore chiral invariance in the $M = 0$ limit.

To $H_{\rm LC}$ we must add an effective interaction, modeled on the missing fermion Z graph, to accomplish cancellation of an infrared singularity in the instantaneous fermion interaction. The singularity occurs when the longitudinal momentum of the instantaneous fermion approaches zero. The effective interaction is constructed from the pair creation and annihilation terms in the Yukawa light-cone energy operator[6]

$$\begin{aligned}\mathcal{P}^-_{\rm pair} &= \frac{g}{2L_\perp\sqrt{L}} \sum_{\underline{pq}si} \left[\frac{\vec{\epsilon}_{-2s}\cdot\vec{p}_\perp}{p^+\sqrt{q^+}} + \frac{\vec{\epsilon}^{\,*}_{2s}\cdot(\vec{q}_\perp - \vec{p}_\perp)}{(q^+-p^+)\sqrt{q^+}} \right] \xi_i b^\dagger_{\underline{p},s} d^\dagger_{\underline{q}-\underline{p},s} a_{i\underline{q}} + \text{h.c.} \\ &\quad + \frac{Mg}{2L_\perp\sqrt{2L}} \sum_{\underline{pq}si} \left[\frac{1}{p^+\sqrt{q^+}} - \frac{1}{(q^+-p^+)\sqrt{q^+}} \right] \xi_i b^\dagger_{\underline{p},s} d^\dagger_{\underline{q}-\underline{p},-s} a_{i\underline{q}} + \text{h.c.}\,,\end{aligned} \tag{5}$$

combined with the denominator for the intermediate state

$$\frac{M^2}{P^+} - p^-_{\rm spectators} - \frac{M^2 + p'^2_\perp}{p'^+} - \frac{M^2 + (\vec{q}'_\perp - \vec{p}_\perp)^2}{q'^+ - p^+} - \frac{M^2 + p_\perp^2}{p^+}\,. \tag{6}$$

To complete the cancellation of the singularity in the kinematic regime of positive longitudinal fermion momentum, the instantaneous interaction is kept only if the corresponding crossed boson graph is permitted by the numerical cutoffs.

The single-fermion eigenstate of the Hamiltonian is written

$$\begin{aligned}\Phi_\sigma &= \sqrt{16\pi^3 P^+} \prod_i \sum_{n_i} \int \frac{dp^+ d^2 p_\perp}{\sqrt{16\pi^3 p^+}} \prod_i \prod_{j_i=1}^{n_i} \int \frac{dq^+_{j_i} d^2 q_{\perp j_i}}{\sqrt{16\pi^3 q^+_{j_i}}} \sum_s \\ &\quad \times \delta(\underline{P} - \underline{p} - \sum_i \sum_{j_i}^{n_i} \underline{q}_{j_i}) \phi^{(n_i)}_{\sigma s}(\underline{q}_{j_i};\underline{p}) \frac{1}{\sqrt{\prod_i n_i!}} b^\dagger_{\underline{p}s} \prod_i \prod_{j_i}^{n_i} a^\dagger_{i\underline{q}_{j_i}} |0\rangle\,,\end{aligned} \tag{7}$$

with normalization

$$\Phi_\sigma'^\dagger \cdot \Phi_\sigma = 16\pi^3 P^+ \delta(\underline{P}' - \underline{P})\,. \tag{8}$$

The wave functions $\phi_{\sigma s}^{(n_i)}$ describe a contribution with n_0 physical bosons and n_i bosons of the ith PV flavor.

Mass renormalization is carried out by fixing the mass M^2 of the dressed single-fermion state. This is imposed by rearranging the eigenvalue problem $H_{\rm LC}\Phi_\sigma = M^2\Phi_\sigma$ into an eigenvalue problem for δM^2:

$$\begin{aligned} x\left[M^2 - \frac{M^2+p_\perp^2}{x} - \sum_i \frac{\mu_i^2 + q_{\perp i}^2}{y_i}\right]\tilde\phi & \\ - \int \prod_j dy_j' d^2 q_{\perp j}' \sqrt{xx'}\mathcal{K}\tilde\phi' &= \delta M^2 \tilde\phi\,, \end{aligned} \tag{9}$$

where the new wave functions are related to the originals by $\tilde\phi = \phi/\sqrt{x}$ and $\mathcal{K}$ is the interaction kernel.

To fix the coupling we set a value for the expectation value $\langle{:}\phi^2(0){:}\rangle \equiv \Phi_\sigma^\dagger {:}\phi^2(0){:}\,\Phi_\sigma$ for the boson field operator ϕ. From a numerical solution it can be computed fairly efficiently in a sum similar to a normalization sum

$$\begin{aligned} \langle{:}\phi^2(0){:}\rangle = \prod_i \sum_{n_i=0} \prod_i \prod_{j_i}^{n_i} \int dq_{j_i}^+ d^2 q_{\perp j_i} \sum_s & \\ \times \left(\sum_{k=1}^n \frac{2}{q_k^+/P^+}\right) \left|\phi_{\sigma s}^{(n_i)}(\underline{q}_{j_i}; \underline{P} - \prod_i \sum_{j_i} \underline{q}_{j_i})\right|^2 &. \end{aligned} \tag{10}$$

The constraint on $\langle{:}\phi^2(0){:}\rangle$ is satisfied by solving it simultaneously with the eigenvalue problem.

3 Diagonalization method

The method used for diagonalization of the eigenvalue problem is based on a special form of the Lanczos algorithm [8] designed to handle the indefinite norm. Let η represent the metric signature, so that numerical dot products are written $\langle\phi'|\phi\rangle = \vec\phi'^* \cdot \eta\vec\phi$. An operator H is self-adjoint with respect to this metric if [9] $\bar H \equiv \eta^{-1} H^\dagger \eta = H$. The Lanczos algorithm for the diagonalization of H then takes the form

$$\begin{aligned} \vec q_{j+1} &= \vec r_j/\beta_j\,, \quad \vec r_j = H\vec q_j - \gamma_{j-1}\vec q_{j-1} - \alpha_j \vec q_j \\ \nu_{j+1} &= \mathrm{sign}(\vec r_j^{\,*}\cdot\eta\vec r_j)\,, \quad \nu_1 = \mathrm{sign}(\vec q_1^{\,*}\cdot\eta\vec q_1)\,, \end{aligned} \tag{11}$$

$$\alpha_j = \nu_j \vec{q}_j^{\,*} \cdot \eta H \vec{q}_j \,, \;\; \beta_j = +\sqrt{|\vec{r}_j^{\,*} \cdot \eta \vec{r}_j|} \,, \;\; \gamma_j = \nu_{j+1}\nu_j\beta_j \,.$$

The original (large) matrix H acquires a new (small) matrix representation T with respect to the basis formed by the vectors $\vec{q}_j$:

$$H \rightarrow T \equiv \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 \dots \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & 0 \dots \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 \dots \\ 0 & 0 & \gamma_3 & . & . \dots \\ 0 & 0 & 0 & . & . \dots \\ . & . & . & . & . \dots \end{pmatrix} . \tag{12}$$

The elements of T are real, and the matrix is self-adjoint with respect to the induced metric ν. We can solve $T\vec{c}_i = \lambda_i \vec{c}_i$ for eigenvalues and right eigenvectors and find that $H\vec{\phi}_i \simeq \lambda_i \vec{\phi}_i$, with

$$\vec{\phi}_i = \sum_k (c_i)_k \vec{q}_k \,, \;\; \vec{\phi}_i^{\,*} \cdot \eta \vec{\phi}_j = \vec{c}_i^{\,*} \cdot \nu \vec{c}_j \,. \tag{13}$$

Extreme eigenvalues are well approximated after only a few iterations.

Because of the indefinite metric, the physical one-fermion state is not necessarily the lowest mass state. It is instead identified by the following characteristics: a positive norm, a real eigenvalue, and the largest bare fermion probability between 0 and 1. Each of these characteristics can be computed without constructing the full eigenvector, provided one saves the first component of each Lanczos vector $\vec{q}_j$ to reconstruct the bare fermion probability.

4 Future work

Current work on this formulation of Yukawa theory is focused on the tuning of DLCQ weighting factors, [3] both in general and specifically with respect to the infrared cancellations between instantaneous fermion interactions, crossed boson graphs, and the effective Z interaction. The singular interactions make the numerical representation more sensitive to the weighting than was the case for earlier model calculations. [3,4]

Within the present no-pair approximation, we can next consider two-fermion states. This would allow consideration of a true bound state. [10] For the full theory, with fermion pair creation, we can again consider the one-fermion and two-fermion sectors. This will require further analysis of the infinities of the theory and determination of the appropriate PV particle types and interactions.

Other theories that could be considered include quantum electrodynamics, where one might use PV regularization to repeat a calculation by Hiller and Brodsky [11] of the electron's anomalous moment for large coupling. Quantum chromodynamics will require a somewhat different approach; a broken supersymmetric formulation, with massive partners playing the role of PV particles may be the correct route.

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